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Conically similar viscous flows. Part 2. One-parameter swirl-free flows

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It was shown in Part 1 of this series that in swirl-free flow there are three different types of axial causes of steady conically similar viscous flow. The three corresponding swirl-free one-parameter families of exact solutions to the Navier–Stokes equations are presented and analysed here in terms of the basic conservation principles for volume and ring circulation. The simplest is the irrotational flow generated by a uniform distribution of volume sources along a half-axis. A second, independent, one-parameter family of solutions is provided by Landau's (1943) solution, where the second moment of ring circulation about the axis is produced at the origin at a finite constant rate. Fresh insight into the nature of this flow is gained by separating and comparing the roles of the diffusion and convection terms in the flux vector for ring circulation. A similar analysis is applied to the remaining independent one-parameter family caused by an antisymmetric (about the origin) conically similar axial distribution of the singularity in Landau's solution. This simple new family of exact solutions is characterized by opposed jets neighbouring the axis of symmetry. When the axial jets are directed inwards, they always erupt into an emergent axisymmetric jet normal to the axis of symmetry. Solutions fail to exist, however, for sufficiently strong axial jets directed outwards.

1. Introduction

It was shown in Part 1 of this series of papers (Pillow & Paull 1985, hereinafter referred to as I) that the swirl-free axisymmetric flow of an incompressible viscous fluid is governed by two basic conservation principles – conservation of volume and conservation of ring circulation. In such flows, in terms of cylindrical polar coordinates (x, σ, ϕ), the velocity field $\boldsymbol{u}(\boldsymbol{r}, t)$ can be described in terms of a scalar stream function ψ in the form

$$\boldsymbol{u} = \frac{1}{\sigma}[\psi_\sigma \hat{\boldsymbol{x}} - \psi_x \hat{\boldsymbol{\sigma}}]. \tag{1.1}$$

In swirl-free axisymmetric flow the vorticity $\boldsymbol{\omega}$ (i.e. curl $\boldsymbol{u}$) is purely azimuthal, and is given by

$$\boldsymbol{\omega} = \sigma l \hat{\boldsymbol{\phi}}, \tag{1.2}$$

where $l/2\pi$ is the ring-circulation volume density. Conservation of ring circulation requires (I (2.3)) that

$$\frac{\partial l}{\partial t} + \operatorname{div} \boldsymbol{J} = -4\pi\nu l \delta(\boldsymbol{\sigma}), \tag{1.3}$$

where (I (2.4))

$$\boldsymbol{J} = l\boldsymbol{u} - 2l\boldsymbol{q}_0 - \nu \nabla l \tag{1.4}$$

and (I (2.5))

$$\boldsymbol{q}_0 = \frac{\nu\hat{\boldsymbol{\sigma}}}{\sigma}, \quad \operatorname{div} \boldsymbol{q}_0 = 2\pi\nu\delta(\boldsymbol{\sigma}). \tag{1.5}$$

(Here I (2.5), for example, refers to equation (2.5) of Part 1.) In terms of ψ, l is given by I (1.14) in the form

$$\psi_{xx} - \frac{1}{\sigma}\psi_{\sigma} + \psi_{\sigma\sigma} = -\sigma^2 l. \tag{1.6}$$

Conically similar viscous flows were defined in I as the special class of axisymmetric flows in which the parameters characterizing the flow causes are of the same dimensions as powers of the kinematic viscosity ν. For such flows, dimensional arguments require that ψ and l take the form $\nu r f(\mu)$ and $\nu g(\mu)/r^3$ respectively (I (1.15), (1.16)), where (r,θ) are polar coordinates in an axial half-plane and $\cos\theta = \mu = x/r$. It follows from (1.6) that $g = -f''$. In steady swirl-free conically similar flow, replacement of g by $-f''$ in I (3.5) requires, after three integrations, that f satisfy (I (4.1))

$$(1-\mu^2)f' + 2\mu f - \tfrac{1}{2}f^2 = A\mu^2 + B\mu + C, \tag{1.7}$$

where A, B and C are constants, which can be determined in terms of the strengths of the basic flow causes.

It was shown in §4 of I that the solutions of (1.7) can be characterized quite simply in terms of four independent axial causes. This present paper analyses in detail the three non-redundant and essentially different types of one-parameter swirl-free conically similar viscous flows that result when all but one of the independent axial causes are eliminated.

This set contains a new family of exact nonlinear Navier–Stokes solutions generated by an antisymmetric distribution of axial component of moment of whirl sources about the origin. The family describes two opposed jets directed along the axis of symmetry. When the jets are directed inwards, the outer flow is parallel to the axis of symmetry and discharges into an axisymmetric radial jet, normal to this axis. The radial jet erupts from the origin, where the axial jets collide. When the opposed jets are directed outwards, the outer flow is directed normal to the axis of symmetry, prior to its entrainment by the jets. As the strength of these jets increases, ring circulation, generated on the axis, is confined to smaller and smaller conical neighbourhoods of the axis. Unbounded solutions finally result once a critical strength is reached. These jet flows are investigated analytically in §4 along with their asymptotic behaviours for strengths that are large, small and just subcritical.

The flows generated by a uniform half-line source of volume and a point source of axial component of moment of whirl complete the set of basic one-parameter swirl-free conically similar viscous flows in accord with the classification provided in §4 of I. The first of these flows is a simple irrotational flow, and is described in §2. Slezkin (1934), Landau (1944) and Squire (1951) have investigated the other flow dynamically and have shown that it is generated by a point source of axial momentum at the origin. The kinematic properties of this flow are investigated in detail in §3 along with the asymptotic behaviour of the flow for both large and small strength of the point cause at the origin.

Throughout §§3 and 4 the conservation principles, for ring circulation and for the axial component of moment of whirl, provide a more detailed understanding of the local balances in the flow. The flux lines for ring circulation also help to reveal the relative importance of the convection, viscous diffusion and viscous convection terms in its flux vector $\boldsymbol{J}$ when they are considered along with the streamlines of the flow.

In all three of the one-parameter problems studied, computer-generated solutions and plotted flux lines illustrate and support the analytic and asymptotic results detailed.

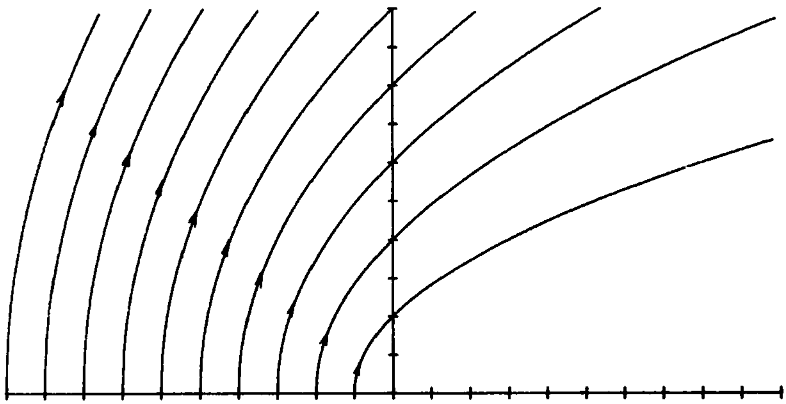

FIGURE 1. The axial half-plane streamlines for a uniform half-line source of mass.

2. The conically similar flow generated by a uniform half-line volume source

Since there is no production of ring circulation, this one-parameter flow is irrotational and $g(\mu) \equiv 0$. It follows that $f(\mu)$ is linear in μ, i.e.

$$f(\mu) = f(1)\left(\frac{1+\mu}{2}\right) + f(-1)\left(\frac{1-\mu}{2}\right). \tag{2.1}$$

If there is only a uniform half-line volume source on the left half-axis of symmetry, then $f(1) = 0$ and

$$\psi = \frac{M_{-1}}{2\pi} r\left(\frac{1-\mu}{2}\right), \tag{2.2}$$

where M_{-1} is the constant line density of volume sources on this half-axis. Consequently the axial half-plane streamlines (figure 1) form a family of confocal parabolas with vertices on the left half-axis of symmetry and with common focus at the origin. A change from source to sink flow (i.e. M_{-1} changes from positive to negative) merely results in a reversal of the direction of flow, since there is no nonlinearity in irrotational flow.

3. The conically similar flow generated by a point source of the axial component of moment of whirl

This section presents the flow produced by a point source of the axial component of moment of whirl, or equivalently by a point source of axial momentum (a point force) (Landau 1944; Squire 1951). An understanding of the flow generated by this cause provides a necessary component for the synthesis of induced axial half-plane flows which arise when conically similar causes are coupled. The dynamics of Landau's flow have been discussed by Batchelor (1967). In this paper the kinematic viewpoint is emphasized in order to elucidate the role of each term in the ring-circulation flux vector. Whilst the rate of production of ring circulation is infinite at the origin, the second moment about the axis of symmetry of this production is finite. This strength L characterizes the flow and is measured by the axial component of moment of whirl emitted radially from the origin.

In the one-parameter flow caused solely by a point source of the axial component of moment of whirl at the origin, there is no uniform production of fluid volume along the axis of symmetry and no antisymmetric distribution about the origin of moment-of-whirl sources. Hence by I (4.4), I (4.5) and I (4.22),

$$A+C = B = A-C = 0. \tag{3.1}$$

The non-dimensional stream function $f(\mu)$ governing the flow is thus the solution of the simple Riccati equation

$$(1-\mu^2)f'+2\mu f-\tfrac{1}{2}f^2 = 0, \tag{3.2}$$

and is

$$f(\mu) = \frac{-2(1-\mu^2)}{\mu-1-2/H} \quad \text{for } H\in(-1,\infty). \tag{3.3}$$

This is the solution given by Landau (1944), Squire (1951) and others. If the edge of the jet is taken to be where $f'(\mu_{\mathrm{E}}) = 0$, then

$$\mu_{\mathrm{E}} = 1+\frac{2}{H}-2\,\mathrm{sgn}\,(H)\left[\frac{1}{H}\left(1+\frac{1}{H}\right)\right]^{\frac{1}{2}}. \tag{3.4}$$

The solution $f(\mu)$ is a monotonic increasing function of H (for any fixed μ), and has, for $\mu\in[-1,1]$,

$$0 \leqslant f(\mu,H) \leqslant 2(1+\mu) \quad (0\leqslant H<\infty) \tag{3.5}$$

and

$$-2(1-\mu) \leqslant f(\mu,H) \leqslant 0 \quad (-1<H\leqslant 0), \tag{3.6}$$

where the linear bounds are attained when $H\to\infty$ and $H\to-1^+$ respectively for $\mu\in(-1,1)$. The strength L of the point source of the axial component of moment of whirl is a monotonic increasing function of H, since I (4.19), when rewritten in the form

$$L = 2\pi\nu^2\int_{-1}^{1}\left[\left(3+\frac{2\mu f}{1-\mu^2}\right)(2-f')f-\frac{\mu f^2}{(1-\mu^2)}\right]\mathrm{d}\mu, \tag{3.7}$$

indicates that

$$\begin{aligned}\frac{\mathrm{d}L}{\mathrm{d}H} &= 4\pi\nu^2\int_{-1}^{1}\left[(1+\mu^2)\left(\frac{f}{1-\mu^2}\right)^2+3\mu\left(\frac{f}{1-\mu^2}\right)+3\right]\frac{\partial f}{\partial H}\,\mathrm{d}\mu, \\ &> 0.\end{aligned} \tag{3.8}$$

The solution $f(\mu)$ thus increases with L. In the limit as $H\to\infty$ or $H\to-1^+$, $L\to\infty$ and $L\to-\infty$ respectively. For any given point source strength L of production of the axial component of moment of whirl at the origin, the solution $f(\mu)$ is thus unique. The simple nature of the solution (3.3) allows the function $L(H)$ to be calculated in the closed form

$$L(H) = 4\pi\nu^2\left\{\frac{8H(H+2)}{3(H+1)}-4\,\frac{(H+2)^2}{H^2}\ln(1+H)+\frac{8(2+H)}{H}\right\}. \tag{3.9}$$

These results indicate that values of H satisfying $-1<H<0$ correspond to negative values of L. The form of (3.3) shows that these solutions can be placed in one-to-one correspondence with those for $H>0$. Values of L of opposite sign then correspond to flows that are reflections of one another in the plane $\mu=0$ ($f(\mu,L) = -f(-\mu,-L)$). Without loss of generality, further discussion is restricted to $L\geqslant 0$.

The solution $f(\mu,L)$ ($L\neq 0$) always has a single internal maximum, and the non-dimensional ring-circulation density $g(\mu)$ is always positive and concentrated in the region of outflow. The derivative of $f(\mu)$ (the non-dimensional radial velocity) is zero at that μ where $f(\mu,L) = 4\mu$, and no inflexions in f or its derivatives occur.

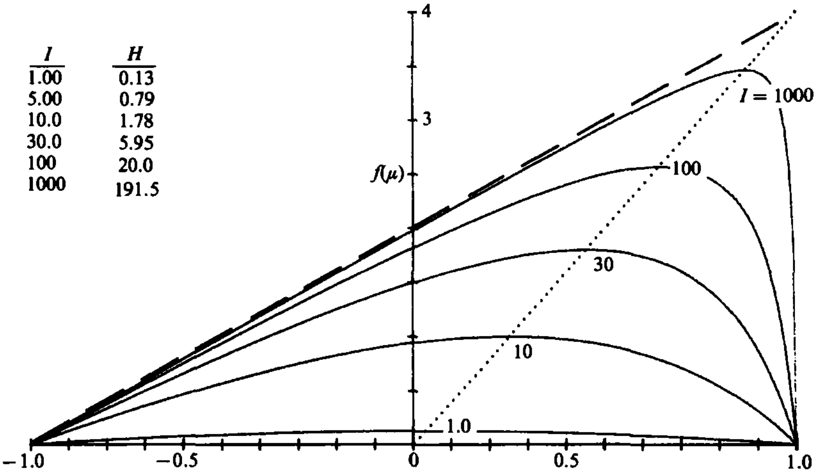

FIGURE 2. The non-dimensional stream function f plotted for various strengths $L = 2\pi\nu^2 I$ of the point source of axial component of moment of whirl.

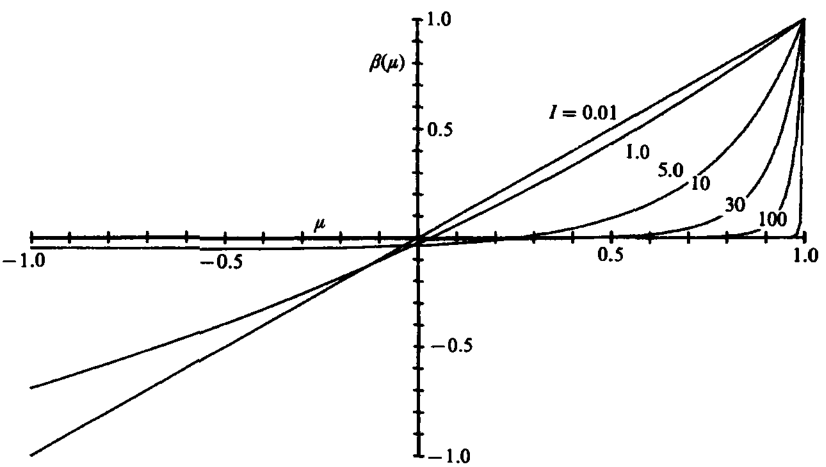

FIGURE 3. The non-dimensional Bernoulli function plotted for the same strengths L as in figure 2.

When the strength L of production is small the solution (3.3) may be expressed as a convergent series in powers of I, where $I = L/2\pi\nu^2$. The differential equation (3.2) then leads to

$$f(\mu) = \tfrac{1}{8}I(1-\mu^2) + \tfrac{1}{128}I^2\mu(1-\mu^2) + \tfrac{1}{1024}I^3(1-\mu^2)(\mu^2-\tfrac{17}{16}) + O(I^4). \tag{3.10}$$

Here the first-order term is the Stokes-flow solution (convection negligible). The second-order term represents the correction to the Stokes-flow solution arising from the induced convection field of the Stokes flow itself.

When L is large the discontinuous limit in (3.3) as $L \to \infty$ ($f(1) = 0$, yet the function $f(\mu, L) \sim 2(1+\mu)$ on $(-1, 1)$) indicates the presence of a developing boundary layer. Inner and outer solutions are then necessary to describe the solution. The integral (3.7) and the differential equation (3.2) show that the outer solution

$$f_{\text{out}}(\mu) \sim 2(1+\mu) - \frac{64}{3}\,\frac{(1+\mu)}{(1-\mu)}\left\{\frac{1}{I} - 8\,\frac{\ln I}{I^2}\right\} + \dots \tag{3.11}$$

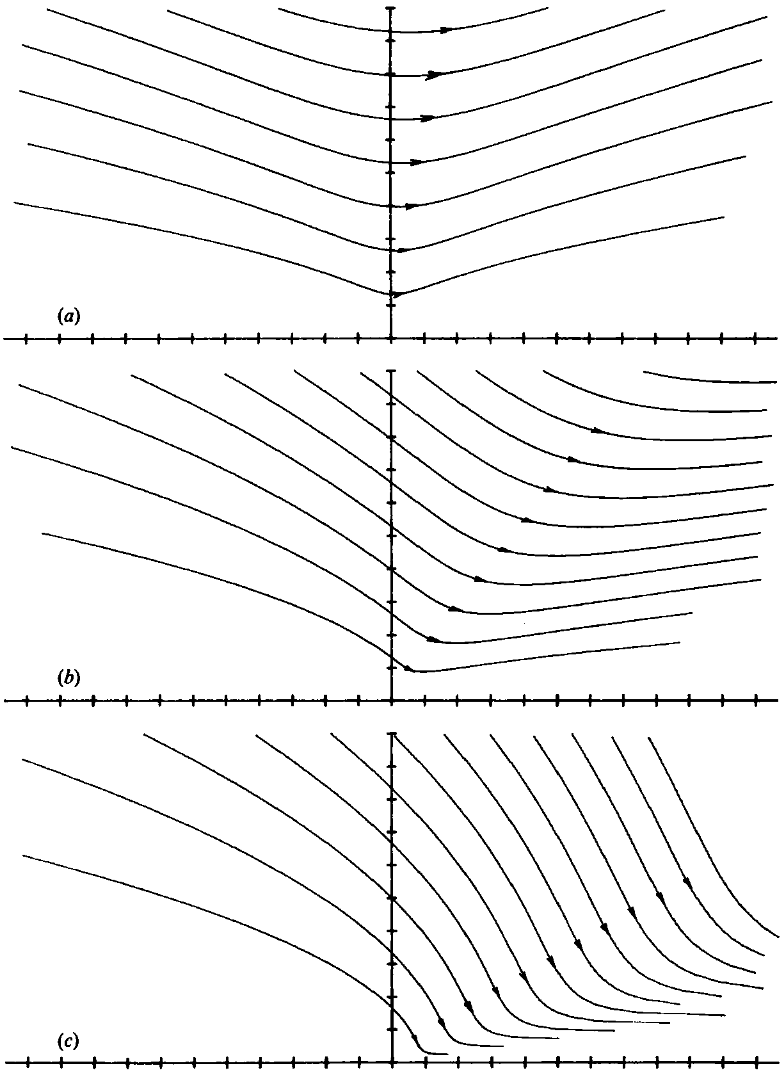

FIGURE 4. The axial, half-plane streamlines corresponding to three typical strengths L of the point source of the axial component of moment of whirl: (a) $L/2\pi\nu^2 = 1$; (b) 30; (c) 1000.

then matches with the expansion in the terminating region,

$$f_{\text{ter}} \sim \frac{4\xi}{\xi+\frac{32}{3}} + \frac{1024}{3}\frac{\ln I}{I}\frac{\xi}{(\xi+\frac{32}{3})^2} + \dots, \tag{3.12}$$

where $\mu = 1-\xi/I$ and $L = 2\pi\nu^2 I$.

For small L, viscous diffusion and convection are the dominant terms in the flux vector for ring circulation. They act so as to spread its density g uniformly over spheres centred on the point source at the origin, (3.10). Radially, the ring circulation density falls off like r^{-3}. This induces a weak convection field which shifts ring circulation from the region of inflow to the region of outflow ($f' = -\frac{1}{4}I\mu$ and $g = \frac{1}{4}I + \frac{3}{64}I^2\mu$). In this way a second-order amount of ring circulation is concentrated in the neighbourhood of the jet. This effect is accentuated as L increases, since g' is positive. When L is large, g is effectively zero in the outer flow (3.11), and, to dominant

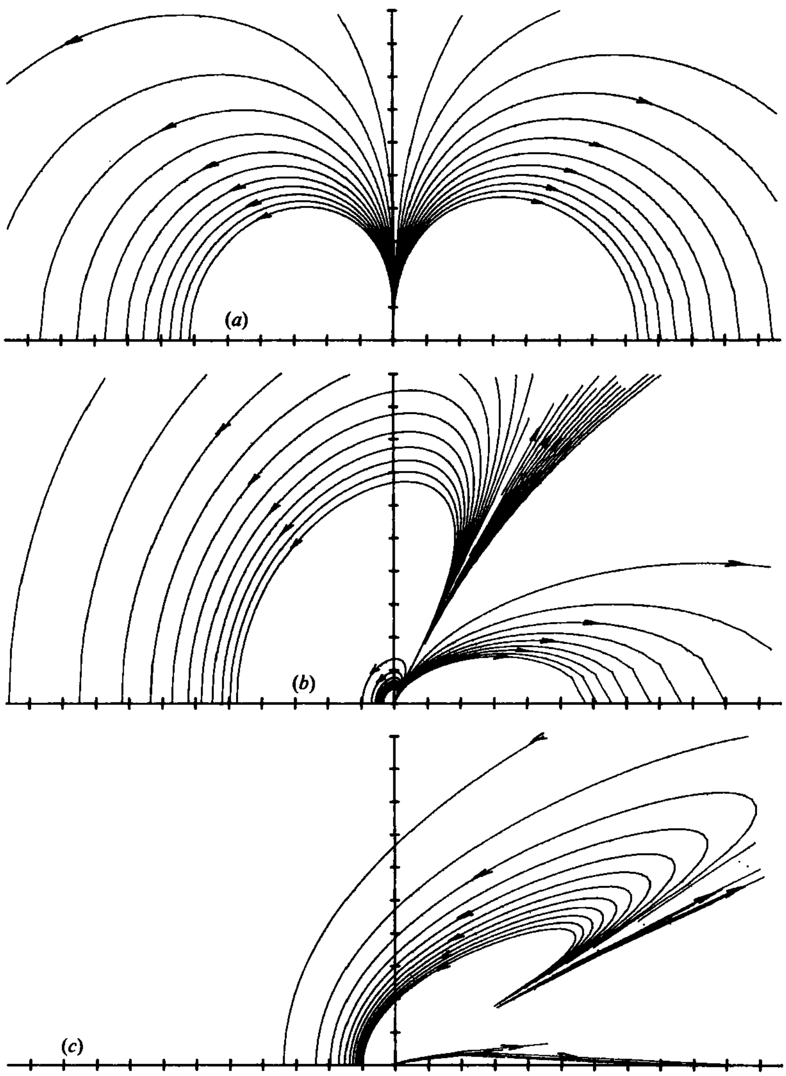

FIGURE 5. The lines of flow of ring circulation in the axial half-plane for the flows illustrated in figure 4: (a) $L/2\pi\nu^2 = 1$; (b) 30; (c) 1000.

order, this flow is irrotational, independent of L and is caused by an apparent uniform volume sink of line density $8\pi\nu$ on the right half-axis of symmetry where $\mu = 1$. This flow describes the outer effects of entrainment in the jet. There is thus a limit to the rate at which the jet can entrain fluid. The terminating solution reveals that the jet region lies within a cone with vertex angle $O(I^{-\frac{1}{2}})$ about the right half-axis. The radial velocity there is $O(I)$, so that, for large I, the amount of fluid entrained by the jet is independent of I, as (3.11) requires. Within the jet, radial convection of ring circulation makes the dominant contribution to the rate L at which the axial component of moment of whirl is being discharged.

The streamlines in the axial half-plane for flows in which L is small, moderate and large appear in figure 4. These correspond to some of the functions $f(\mu)$ plotted in figure 2.

Further information about the movement of ring circulation is provided by its lines of flow in an axial half-plane. Along these lines its flux function, the Bernoulli function I (2.14) and I (3.16), is constant. The petal-shaped flow lines of figure 5 are typical examples. The nondimensional Bernoulli function $\beta(\mu)$ is plotted in figure 3. The function $\beta(\mu)$ always has an internal zero, and when $f'(-1) > 1$ an internal minimum for $\beta(\mu)$ occurs.

The zero of the Bernoulli function occurs when $f' = 0$. This serves to define the edge of the jet by indicating the direction of the separatrix in the flow of ring circulation. In this direction the radial velocity of the fluid is in fact zero, and ring circulation is carried away from the singularity at the origin by the dominance of viscous diffusion over viscous convection of ring circulation. Transversely, the effects of velocity and viscous convection just balance the diffusion of ring circulation. Away from this cone, viscous convection eventually sweeps ring circulation towards the axis of symmetry, where it is destroyed by the sink on the right-hand side of I (2.3). This accounts for the total flux of ring circulation into the axis of symmetry, since there is no gradient for ring circulation to diffuse down normal to this axis, and there are no line sources of volume present that could provide transverse convection. The rate of production of ring circulation at the origin is infinite, but it is totally absorbed on the two half-axes of symmetry, since the sink-strength line density there becomes infinite like $1/r^3$ as the origin is approached. The singularity of ring circulation at the origin is characterized by the finite size of production L of the axial component of moment of whirl.

In Stokes flow (induced self-convection negligible) ring circulation behaves like heat diffusing with diffusivity ν in the forced convection field of a uniform volume sink on the axis with line density $4\pi\nu$ (figure 5*a*). There is a singularity of heat locally resembling a quadrupole at the origin, and all heat is completely removed as it flows into the half-axes by line sinks there of line density $4\pi\nu^2 g(\pm 1)/|x|^3$. The second-order induced convective flux of ring circulation causes the cone of transverse balance in this flow field to tilt over as L increases (figure 5*b*). When the axial component of moment of whirl production is large, the jet lies within a cone with vertex angle $O(I^{-\frac{1}{2}})$ (figure 5*c*). Away from this cone the ring-circulation density is $O(I^{-1})$, compared with a density of order I^2 within the jet.

4. The flow generated by an antisymmetric conically similar distribution about the origin of sources of the axial component of moment of whirl

The remaining independent swirl-free axial cause of conically similar viscous flow provided by the classification in §4 of I is the antisymmetric distribution about the origin of sources of the axial component of moment of whirl (or axial momentum) with line density inversely proportional to the distance from the origin. The one-parameter flow associated with this cause has not previously been investigated, although it has played a role in some of the solutions obtained previously (e.g. Serrin's (1972) half-space flow with swirl). As with Landau's flow (§3), this new flow is associated with conically similar jets. In Landau's case there is a single jet directed parallel to the axis of symmetry. Here opposed jets appear about the axis of symmetry in accordance with the odd distribution about the origin of Landau's singularity.

The line density of the distributed production along the axis of the axial component of moment of whirl is K/x. For negative K the jets along the axis generated by this ring-circulation production are directed inwards. (Since K/x is negative on the right half-axis if K is negative, this is in qualitative agreement with the results of §3, where negative production of the axial component of moment of whirl at the origin (i.e. $L < 0$) led to a jet directed towards the left.) These opposed jets erupt into a radial outflow neighbouring the plane through the origin and normal to the axis of symmetry. As K increases in magnitude, this jet becomes more developed. However, ring circulation in these circumstances is not confined to a small neighbourhood of the ring-circulation causes along the axis of symmetry, since the induced convection field, in being parallel to the axis of symmetry, does not oppose ring circulation diffusing against the fictitious convection field. Away from the driving cause and the jet normal to the axis of symmetry, the flow is not irrotational, but to first order is conically similar flow parallel to the axis of symmetry inwards towards the normal jet. For positive K the sense of the flow reverses and convection now opposes diffusion away from the driving singularity. As K increases, the outward jets surrounding the axis of symmetry narrow in width. Convection and diffusion of ring circulation in the normal jet are now in unison, and this jet now spreads out to fill almost the whole space. The outer flow thus becomes more irrotational in nature until, when $K = 32\pi\nu^2$, the convection field is sufficiently strong to isolate completely the distributed production of the axial component of moment of whirl on the axis from the outside flow. The (outer) flow is then potential flow towards an apparent uniform line sink of volume with line density $8\pi\nu$. Strong outward-directed jets along the axis of symmetry account for this entrained fluid and herald the non-existence of finite solutions for $K > 32\pi\nu^2$.

The solution for the non-dimensional stream function $f(\mu)$ can be expressed in a closed form involving Legendre functions with the cause strength K specifying their order. Since there are to be no uniform half-line volume sources, the constants $A+C$ and B in I (4.1) are zero by I (4.4) and I (4.5). The governing differential equation is then

$$(1-\mu^2)f'+2\mu f-\tfrac{1}{2}f^2 = C(1-\mu^2), \tag{4.1}$$

where a boundary condition is provided by the requirement that there be zero production L of the axial component of whirl at the origin. After some integration by parts, $L = 0$ in I (4.19) gives

$$0 = \int_{-1}^{1}\left[2\mu(f')^2-\frac{\mu f^2}{1-\mu^2}+6f\right]d\mu. \tag{4.2}$$

This last constraint can be simply satisfied by seeking odd solutions to (4.1) (i.e. requiring $f(0) = 0$). The axial component of moment of whirl emitted from the axis per unit length per unit time is K/x, where

$$K = 8\pi\nu^2 C, \tag{4.3}$$

in accord with I (4.22).

The substitution

$$f = \frac{C\phi}{\phi'}, \tag{4.4}$$

similar to, but different from, the standard Riccati substitution I (3.11), is the most convenient for obtaining the solution for f from (4.1) in closed form. This results in Legendre's equation for ϕ:

$$(1-\mu^2)\phi''-2\mu\phi'+\alpha(\alpha+1)\phi = 0, \tag{4.5}$$

with

$$\phi(0) = 0 \tag{4.6}$$

in order to satisfy (4.2). Here

$$C = 2\alpha(\alpha+1). \tag{4.7}$$

The solution is thus

$$f(\mu) = -2\alpha(\alpha+1)(1-\mu^2)^{\frac{1}{2}} \frac{Q_\alpha(\mu)+\frac{1}{2}\pi \tan(\frac{1}{2}\pi\alpha) P_\alpha(\mu)}{Q'_\alpha(\mu)+\frac{1}{2}\pi \tan(\frac{1}{2}\pi\alpha) P'_\alpha(\mu)}, \tag{4.8}$$

where

$$\alpha = -1+(1+2C)^{\frac{1}{2}}. \tag{4.9}$$

The nature of this solution is best analysed by the application of differential-inequality techniques to the nonlinear Riccati differential equation (4.1) and to its linearized form

$$h''+\frac{C}{1-\mu^2}h = 0, \tag{4.10}$$

with

$$h(0) = 1, \quad h'(0) = 0, \tag{4.11}$$

which results from the standard substitution (I (3.11))

$$f = -2(1-\mu^2)\frac{h'}{h}. \tag{4.12}$$

As C increases from zero, the first eigenfunction with $h(1) = 0$ in (4.10) is $h_1 = 1-\mu^2$, when $C = 4$ and $\alpha = 1$. For $C < 4$ the solution h of (4.10) and (4.11) has $h > 0$ on $[-1, 1]$, indicating f is bounded. Even for $C = 4$, $f(\mu)$ is bounded and equals 4μ. For $C > 4$, h has simple internal zeros on $[-1, 1]$ which increase in number by two each time C exceeds a number of the form $2\alpha(\alpha+1)$, with α an odd integer. Such values of C give rise to unbounded solutions for $f(\mu)$ and are not considered further in this paper.

For $C < 4$ the non-dimensional stream function $f(\mu)$ always has only one internal zero, one maximum and one minimum on $[-1, 1]$. A single separatrix $\mu = 0$ is thus always present in the flow. When $K < 0$ outflow occurs about this separatrix. Inflow about the separatrix occurs for $K > 0$. For $K < 32\pi\nu^2$ ($K = 8\pi\nu^2 C$) the solution $f(\mu)$ has, as $\mu \to \pm 1$,

$$f(\mu) \sim \frac{C}{2}(1-\mu^2)\ln\frac{1+\mu}{1-\mu}, \tag{4.13}$$

$$f'(\mu) \sim -C\mu \ln\frac{1+\mu}{1-\mu} \tag{4.14}$$

and

$$g(\mu) \sim \frac{2C\mu}{1-\mu^2}. \tag{4.15}$$

A logarithmically stronger flow than occurred in §3 is thus induced about the distributed sources of the axial component of moment of whirl along the axis of symmetry.

Asymptotic forms of the solution (4.8) can be obtained for strengths K satisfying $K \ll -32\pi\nu^2$, $K \approx 0$ and $K \approx 32\pi\nu^2$. When K is small the convergent series

$$f = \frac{Cf_0}{1!}+\frac{C^2 f_1}{2!}+\frac{C^3 f_2}{3!}+\dots \tag{4.16}$$

reconstructs the solution, where

$$f_0 = (1-\mu^2)\int_0^\mu \frac{d\xi}{1-\xi^2} = \frac{1-\mu^2}{2}\ln\frac{1+\mu}{1-\mu} \tag{4.17}$$

and

$$f_{m+1} = (1-\mu^2)\int_0^\mu \frac{\left[\sum\limits_{i=0}^{m}\binom{m+1}{i+1} f_i(\xi)\,f_{m-i}(\xi)\right]\mathrm{d}\xi}{(1-\xi^2)^2}. \tag{4.18}$$

The series (4.16) is convergent for $|C| < 4$, and the flow it describes is '*almost-Stokes*' flow when $|C| \ll 4$.

When $K \ll -32\pi\nu^2$ the first-order outer solution of (4.1) is given by

$$f^2_{\mathrm{out}} = -2C(1-\mu^2). \tag{4.19}$$

This outer solution is multivalued and does not satisfy either $f_{\mathrm{out}}(0) = 0$ or the asymptotic behaviour required of the solution as the axis of symmetry is approached. Boundary layers are thus present at $\mu = \pm 1$ and at $\mu = 0$, with the latter being in the form of a transition between the outer branches. The former will be called terminating regions. If

$$\epsilon = (-C)^{-\frac{1}{2}}, \tag{4.20}$$

matched asymptotic expansions representing the solution f are

$$f_{\mathrm{out}} = \frac{1}{\epsilon}\sqrt{2}\,(1-\mu^2)^{\frac{1}{2}} + \mu + \epsilon\,\frac{2+\mu^2}{8(1-\mu^2)} + O(\epsilon^2) \tag{4.21}$$

in the outer region $-1 < \mu < 0$,

$$f_{\mathrm{te}} = 2\xi^{\frac{1}{2}}\frac{K_0(\xi^{\frac{1}{2}})}{K_1(\xi^{\frac{1}{2}})} - \epsilon^2\left\{\xi^{\frac{3}{2}}\frac{K_0(\xi^{\frac{1}{2}})}{K_1(\xi^{\frac{1}{2}})} + \tfrac{1}{6}\xi^3\left[1-\frac{K_2^2(\xi^{\frac{1}{2}})}{K_1^2(\xi^{\frac{1}{2}})}\right]\right\} + O(\epsilon^4), \tag{4.22}$$

where $\mu = -1+\epsilon^2\xi$, in the terminating region (the K_0, K_1 and K_2 are modified Bessel functions), and

$$f_{\mathrm{tr}} = -\frac{\sqrt{2}}{\epsilon}\tanh\left(\frac{\xi}{\sqrt{2}}\right) + \epsilon\left[\xi + \frac{\xi^2-1}{\sqrt{2}}\tanh\left(\frac{\xi}{\sqrt{2}}\right) - \frac{1}{2}\left(\xi+\frac{\xi^3}{3}\right)\operatorname{sech}^2\left(\frac{\xi}{\sqrt{2}}\right)\right] + O(\epsilon^3), \tag{4.23}$$

where $\mu = \epsilon\xi$, in the transition region. The solution $f(\mu)$ is required to be odd about the origin.

The final situation to be discussed occurs when C is large and positive. In this case no outer solution, as such, exists and the terminating expansion about $\mu = -1$ is given in terms of Bessel functions (as opposed to modified Bessel functions). The denominator of the terminating expression then oscillates with increasing frequency as $C \to +\infty$. This culminates in unbounded solutions for f if C becomes too large. Bounded solutions result only when $K \leqslant 32\pi\nu^2$ ($C \leqslant 4$). In the special case when $C = 4$, $f(\mu)$ is bounded but has $f(\pm 1) = \pm 4$. The solution $f(\mu)$ in this case is the solution that results from the first-order Legendre polynomial in (4.8); that is, $f(\mu) = 4\mu$.

The solution corresponding to $C = 4$ fails to satisfy the homogeneous boundary conditions at $\mu = \pm 1$, and indicates the presence of a developing boundary layer as $\epsilon = 4-C$ tends to zero from above (i.e. $\epsilon \to 0^+$). The method of matched asymptotic expansions then yields an outer solution

$$f_{\mathrm{out}} = 4\mu + \epsilon\,\frac{\frac{1}{3}\mu^3-\mu}{1-\mu^2} + \epsilon^2\,\frac{\left(\dfrac{\mu^3}{3} - 4\mu + \dfrac{2\mu}{1-\mu^2} + \ln\dfrac{1+\mu}{1-\mu}\right)}{18(1-\mu^2)} + O(\epsilon^3), \tag{4.24}$$

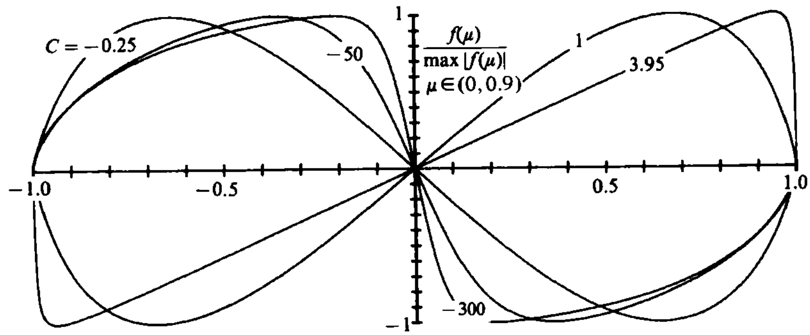

FIGURE 6. The non-dimensional stream function f plotted for various strengths $K = 8\pi\nu^2 C$ of the odd distribution of the axial component of moment of whirl sources.

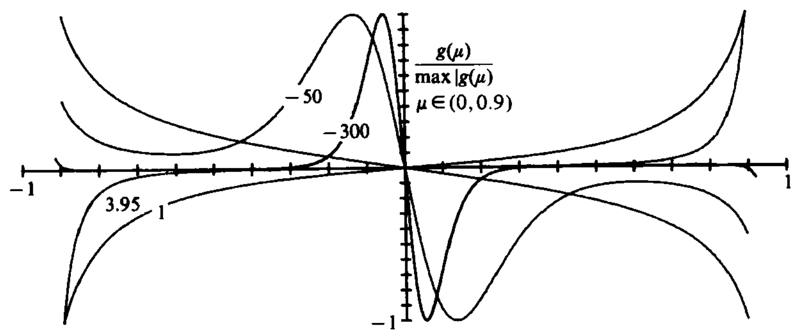

FIGURE 7. The ring-circulation distribution in the flows illustrated in figure 6.

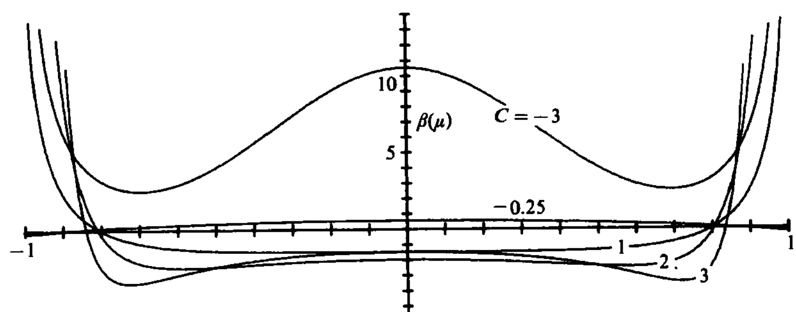

FIGURE 8. The Bernoulli function corresponding to the flows illustrated in figure 6.

while the expansion in the terminating region about $\mu = -1$ (similar to $\mu = +1$) is

$$f_{\mathrm{te}} = \frac{-4\xi}{\xi+B} + \epsilon \ln \epsilon \left\{\frac{4B^2\xi}{(\xi+B)^2}\right\} + \frac{\epsilon\xi}{(\xi+B)^2}\{4\xi^2+10B\xi+4B^2\ln\xi+D\}, \tag{4.25}$$

where $\mu = -1+\epsilon\xi$, $B = \frac{1}{12}$ and $D = \frac{17}{108}$. To first order the outer expansion is irrotational flow towards an apparent uniform line volume sink of strength $8\pi\nu$. This feeds, by entrainment, an opposed pair of axial jets directed outwards from the origin as indicated by (4.25) (cf. (3.12)). Typical non-dimensional stream functions and streamlines appear in figures 6 and 9 respectively.

Whilst this completes the mathematical analysis, the remainder of this section will,

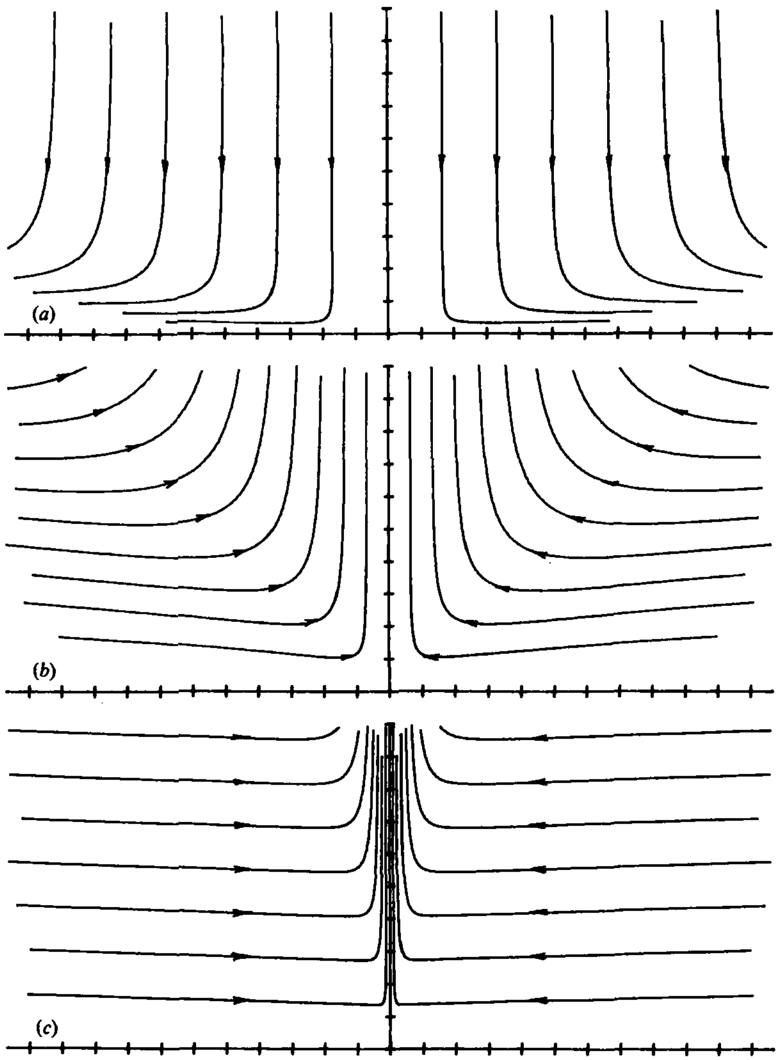

FIGURE 9. Axial half-plane streamlines for some typical examples of the strength $K = 8\pi\nu^2 C$: (a) $K/2\pi\nu^2 = 15.8$; (b) -1; (c) -1200.

for the sake of physical completeness, explore the behaviour, for large and small C, of both the streamlines of volume and the flux lines of circulation, since this throws light on the relative strengths of the individual flux terms in the flux vectors for these conserved quantities. When the strength of the distributed production of the axial component of moment of whirl is small (i.e. $|C| \ll 4$) the power-series solution (4.16) describes the flow. To first order, ring circulation is concentrated in a neighbourhood of the axis of symmetry as a result of viscous convection balancing the normal component of viscous diffusion. In contrast with the problem of §3, convection of ring circulation is not subdominant in all regions of the flow despite $-\frac{1}{2}f^2$ in (4.1) being subdominant to first order. This is a result of the strong induced radial velocity caused by the ring circulation concentrated about the singular source on $|\mu| = 1$. This effect does not occur in the Landau jet of §3, since no physical supply of ring circulation occurs distributed along the axis of symmetry, in that case. The flow here when K is small is thus not everywhere Stokes flow, though that description is valid away from the axis of symmetry. The second-order term of (4.16) indicates that the convection field of the first-order Stokes flow sweeps ring circulation away from a neighbourhood of the axis of symmetry when K is negative and in the opposite sense when K is positive. In this way, for negative K, ring circulation is transported into the jet that erupts from the converging convection fields present about the axis of symmetry. (The normal balance of diffusion and viscous convection of ring circulation, to the highest order, in the neighbourhood of the axis determines the direction of the radial convection field there; that is, $g \sim 2C\mu/(1-\mu^2)$ implies $f' \sim -C\mu \ln[(1+\mu)/(1-\mu)]$.) As K (negative) is decreased, more and more ring circulation is concentrated in the jet region normal to the axis of symmetry until this jet becomes well-developed ($C \ll -4$) and the matched asymptotic expansions (4.20)–(4.23) are valid. In this situation the outer flow is not irrotational flow to first order, but rather conically similar flow parallel to the axis of symmetry inwards towards the jet normal to the axis of symmetry where it is entrained by an apparent conically similar distribution of sinks over the plane $\mu = 0$ with area density $\nu(-8C)^{\frac{1}{2}}/r$. The entrained fluid has radial velocity $O(-C)$ outwards (as given by the transition expansion (4.23)). This conserves volume, since the jet lies within cones whose semivertex angle differs from $\frac{1}{2}\pi$ by $O[(-C)^{\frac{1}{2}}]$.

When K becomes positive the ring-circulation production on the half-axes changes sign, and the induced convection field of the outer flow now opposes the diffusion of ring circulation from the axis of symmetry. As a result of this action the flow becomes almost entirely irrotational and normal to this axis as $K \to 32\pi\nu^2-$. When this value is attained the convection field completely dominates the effects of diffusion of ring circulation, and the (outer) solution is then generated purely by entrainment from an apparent uniform line sink of volume of line density $8\pi\nu$. The entrant jet, normal to axis of symmetry, has now spread out to fill the entire space. Any further decrease in K results in unbounded flows with an increased number of cones on which the solution is unbounded. Such flows are interesting possibly from a mathematical viewpoint and do have applications in certain cone-flow problems with boundaries, but are not considered further here.

The axial half-plane streamlines of three typical flows appear in figure 9. The features noted above are clearly visible in these flows.

The lines of flow for ring circulation are determined by the Bernoulli function $B = \nu^2\beta(\mu)/r^2$, where for the problem of this section

$$\beta(\mu) = -\tfrac{1}{2}[(1-\mu^2)g]' + \tfrac{1}{2}fg, \tag{4.26}$$

$$= \tfrac{1}{2}f'(f'-2) - C \tag{4.27}$$

and
$$\beta'(\mu) = (1-f')g(\mu). \tag{4.28}$$

The flux vector of ring circulation is then

$$\frac{\nu^2}{2\pi r^4}\left[\beta'(\mu)\hat{\mathbf{r}} - \frac{2\beta(\mu)}{(1-\mu^2)^{\frac{1}{2}}}\hat{\boldsymbol{\theta}}\right].$$

The expression (4.26) displays explicitly the viscosity-dependent fluxes (diffusion and viscous convection) grouped into a perfect derivative along with the convective flux, and indicates, since $g \sim 2C\mu/(1-\mu^2)$ as $|\mu| \to 1$, that lateral diffusion of ring circulation away from the distributed sources of the axial component of moment of

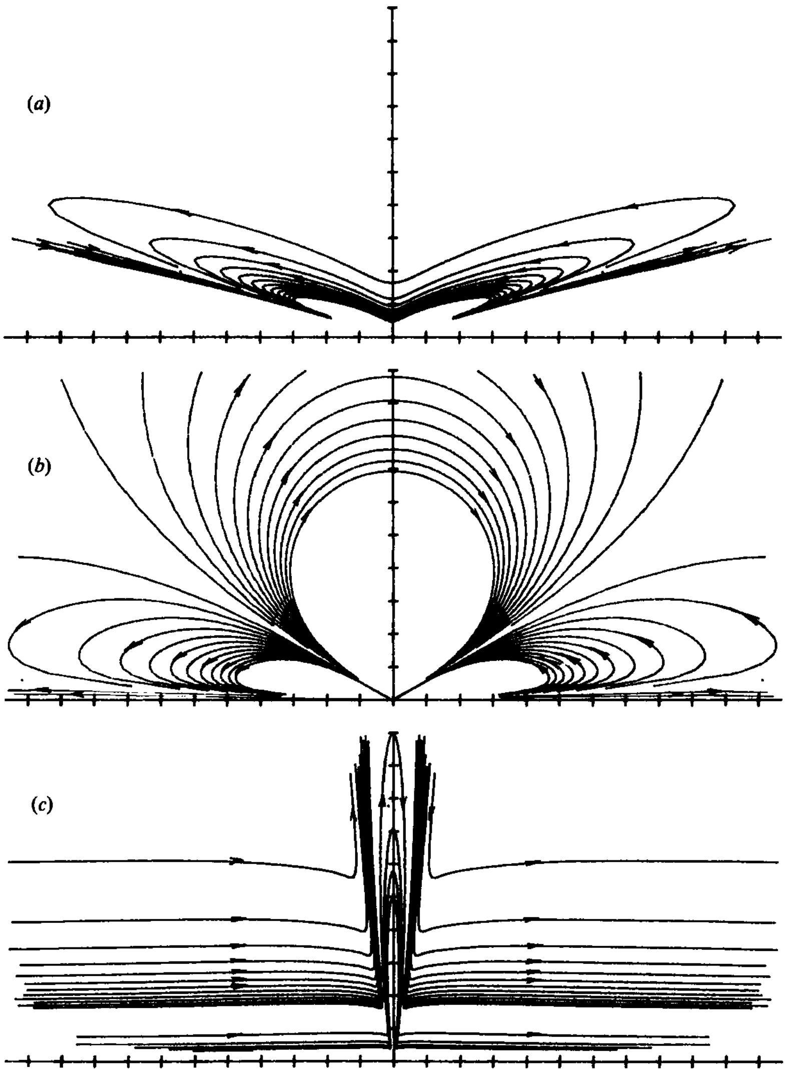

FIGURE 10. Lines of flow of ring circulation when an odd distribution of the axial component of moment of whirl sources is present: (a) $K/2\pi\nu^2 = 15.8$; (b) -1; (c) -1200.

whirl is, to its highest order, balanced by viscous convection. (Convection of ring circulation laterally is subdominant to each of these processes.) This accounts for the large concentration of ring circulation near the axis of symmetry, which would otherwise diffuse outwards. The net rate of ring-circulation production is associated with a lower-order mismatch. The second expression for $\beta(\mu)$ gives the order of the flux of ring circulation at the singularity ($\beta(\mu) \sim \frac{1}{2}C^2 \ln^2(1-\mu)$), though it destroys all meaning of the processes that contribute to this flux. Careful examination of the first expression reveals that it is diffusion slightly overwhelming viscous convection that provides this flow, and not a convective flux as (4.27) might suggest.

Some typical ring-circulation flux lines are shown in figure 10, while the graphs of $\beta(\mu)$ appear in figure 8.

Figure 10 indicates that the ring-circulation flux lines always asymptote to the axis of symmetry and that their sense here does not alter with a change of sign in K. This is the non-Stokes-flow behaviour resulting from radial convection of ring circulation always being significant near the axis of symmetry. The rectification of this flux of ring circulation results from the nonlinearity of the convective flux, while the left–right preference for the axial flow is simply a consequence of the anticlockwise-positive convention for circulation.

A flow that is just subcritical appears in figure 10(*a*), $K \approx 32\pi\nu^2$. As a consequence of an outer potential flow being entrained by the axial jet, the right half of this flow strongly resembles figure 5(*c*) (compare (4.25) with (3.12)) and, except in the immediate neighbourhood of the axis, is qualitatively as described in §3 for large L. In figure 10(*c*) $K \leqslant -32\pi\nu^2$, and convection of the large ring-circulation density by the outer flow inwards parallel to the axis is now the dominant feature. Collision of the outer flows gives rise initially to a radial flux of ring circulation, but none leaks to infinity because of annihilation from the image itself. When almost-Stokes flow occurs for $0 > K \geqslant -4\pi\nu^2$ figure 10(*b*) demonstrates that there exist four separatrices in the flow of ring circulation because of the delicate transverse balances that can develop between convection, viscous convection and diffusion.

REFERENCES

BATCHELOR, G. K. 1967 *An Introduction to Fluid Dynamics*. Cambridge University Press.

LANDAU, L. D. 1944 A new exact solution of the Navier–Stokes equation. *Dokl. Akad. Nauk SSSR* **43**, 286–288.

PILLOW, A. F. & PAULL, R. 1985 Conically similar viscous flows. Part 1. Basic conservation principles and characterization of axial causes in swirl-free flow. *J. Fluid Mech.* **155**, 327–341.

SERRIN, J. 1972 The swirling vortex. *Phil. Trans. R. Soc. Lond.* A **271**, 325–360.

SLEZKIN, M. A. 1934 On a case of integrability of the complete differential equations of a viscous fluid. *Moskov. Gos. Inst. Kul. Učen. Zap.* **2**, 89–90.

SQUIRE, H. B. 1951 The round laminar jet. *Q. J. Mech. Appl. Maths* **4**, 321–329.